

BOUNDEDNESS CRITERIA FOR SINGULAR INTEGRALS AND SQUARE FUNCTIONS

Steve Hofmann

University of Missouri, USA

hofmanns@missouri.edu

Abstract

Singular Integral Operators (“SIOs”) and Square Functions (which may be viewed as vector valued versions of SIOs) arise often in complex analysis and in the theory of partial differential equations. For example, the prototypical SIOs are the Hilbert Transform:

$$Hf(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy,$$

which relates the real and imaginary parts of the boundary values of an analytic function in the upper half plane \mathbb{R}_+^2 , and, in higher dimensions, the Riesz Transforms

$$R_j f(x) := C_n \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy, \quad 1 \leq j \leq n,$$

which relate the normal and tangential components of the boundary values of the gradient of a harmonic function in the half space \mathbb{R}_+^{n+1} . The terminology “singular integral” refers to the fact that the kernels of these integral operators possess a singularity which just fails to be integrable, and therefore the operators must be defined in some limiting or “principal value” (“p.v.”) sense. These prototypical SIOs are of convolution type, so their boundedness on L^2 (which is the fundamental desired property of such operators) may be verified via Plancherel’s Theorem (that is, one may exploit the fact that convolution operators are “diagonalized” by the Fourier transform).

On the other hand, there are many other important examples of SIOs, arising, e.g., in the theory of variable coefficient elliptic PDE, and in the theory of analytic functions in domains with non-smooth boundaries, which are *not* of convolution type, and for these, Plancherel’s Theorem is not available as a tool to establish L^2 boundedness. In this talk, we shall present a survey of progress on the development of criteria to verify the L^2 boundedness of non-convolution SIOs and square functions, and we shall discuss some applications.