CONVERGENCE AND COLLAPSE OF RIEMANNAIN MANIFOLDS

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Notions of convergence play a central role in Riemannian geometry, often having an impact long before and long after their formal introduction.

Around 1980, Gromov started a geometric revolution when he introduced what is now called the Gromov–Hausdorff metric, in part to put Cheeger's 1967 finiteness theorem into a broader context. By definition, the distance between two compact spaces X and Y is $< \varepsilon$ if and only if there is a (not necessarily continuous!) map $f : X \longrightarrow Y$ so that

$$|\mathrm{dist}\,(x_1, x_2) - \mathrm{dist}\,(f\,(x_1)\,, f\,(x_2))| < \varepsilon \text{ for all } x_1, x_2$$

and for all $y \in Y$, there is an $x \in X$ so that

$$|\mathrm{dist}\,(y, f\,(x))| < \varepsilon.$$

Given the very crude nature of Gromov–Hausdorff distance, it is surprising how extremely important it has become. So many famous results rely on the notion of Gromov–Hausdorff convergence that it is not practical to list them; Perelman's Stability Theorem, is just one of many. Gromov–Hausdorff distance has permeated many aspects of geometry, from comparison geometry, to large scale (hyperbolic) geometry, to Kälher geometry, to geometric group theory.

We will explore the impact of Gromov-Hausdorff convergence on manifolds with curvature bounded from below, starting with Gromov and Cheeger's celebrated work, through Perelman's Stability Theorem, and also current open problems about collapse with a lower curvature bound.

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