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2-DIMENSIONAL MINIMAL CONES IN $\mathbb{R}^4$

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Results from V. Feuvrier and XiangYu Liang

Goal of the lecture: first step towards a characterization of 2-dimensional minimal cones in $\mathbb{R}^4$, and description of a result on the almost orthogonal union of two two-planes.

Here minimal will be in the sense of soap films (or Almgren minimal sets), as follows.

Definition. *Let $0 < d < n$ be integers, and $\Omega \subset \mathbb{R}^n$ open. The closed set $E \subset \Omega$ is minimal in Ω when*

$$\mathcal{H}^d(E \setminus F) \leq \mathcal{H}^d(F \setminus E) \tag{1}$$

for all competitors F for E in Ω.

Definition. *A competitor for E in Ω is a set $F = f_1(E)$, where*

$$(x,t) \to f_t(x) : E \times [0,1] \to \Omega \text{ is continuous,} \tag{2}$$

$f_0(x) = x$ *for* $x \in E$, *and, if we set* $W_t = \{x \in E\,;\, f_t(x) \neq x\}$,

$$\bigcup_{0 \leq t \leq 1} [W_t \cup f_t(W_t)] \text{ is relatively compact in } \Omega. \tag{3}$$

We also require f_1 to be Lipschitz.

So F is a deformation of E in Ω.
The condition on the f_t is not needed when Ω is, say, convex.
Important : f_1 is not always injective; we are allowed to pinch.
Almgren's definitions are almost equivalent.
We shall only look at reduced sets: E is equal to the closed support of $\mathcal{H}^2$ restricted to E. Easy reduction

We define almost-minimal sets with the gauge function h (with $\lim_{r\to 0} h(r) = 0$) the same way but we require that

$$\mathcal{H}^d(E \setminus F) \leq \mathcal{H}^d(F \setminus E) + r^d h(r) \tag{4}$$

when $F = f_1(E)$ is a competitor for E in Ω, such that $\bigcup_{0\leq t\leq 1}[W_t \cup f_t(W_t)]$ is contained in a ball of radius r.

We worry about existence and regularity for these sets. Even when $d = 1$, they are not smooth (pictures).

So far, mostly regularity results inside Ω, but no general existence results for Plateau problems, and not much boundary regularity available.

For inside regularity at least, knowing the minimal cones helps a lot, because for all $x \in E$, the density

$$r \to \theta(r) = r^{-d}\mathcal{H}^d(E \cap B(x,r)), \tag{5}$$

is almost monotonous, we have theorems on limits, and every constant-density minimal set (including any blow-up limit) is a cone.

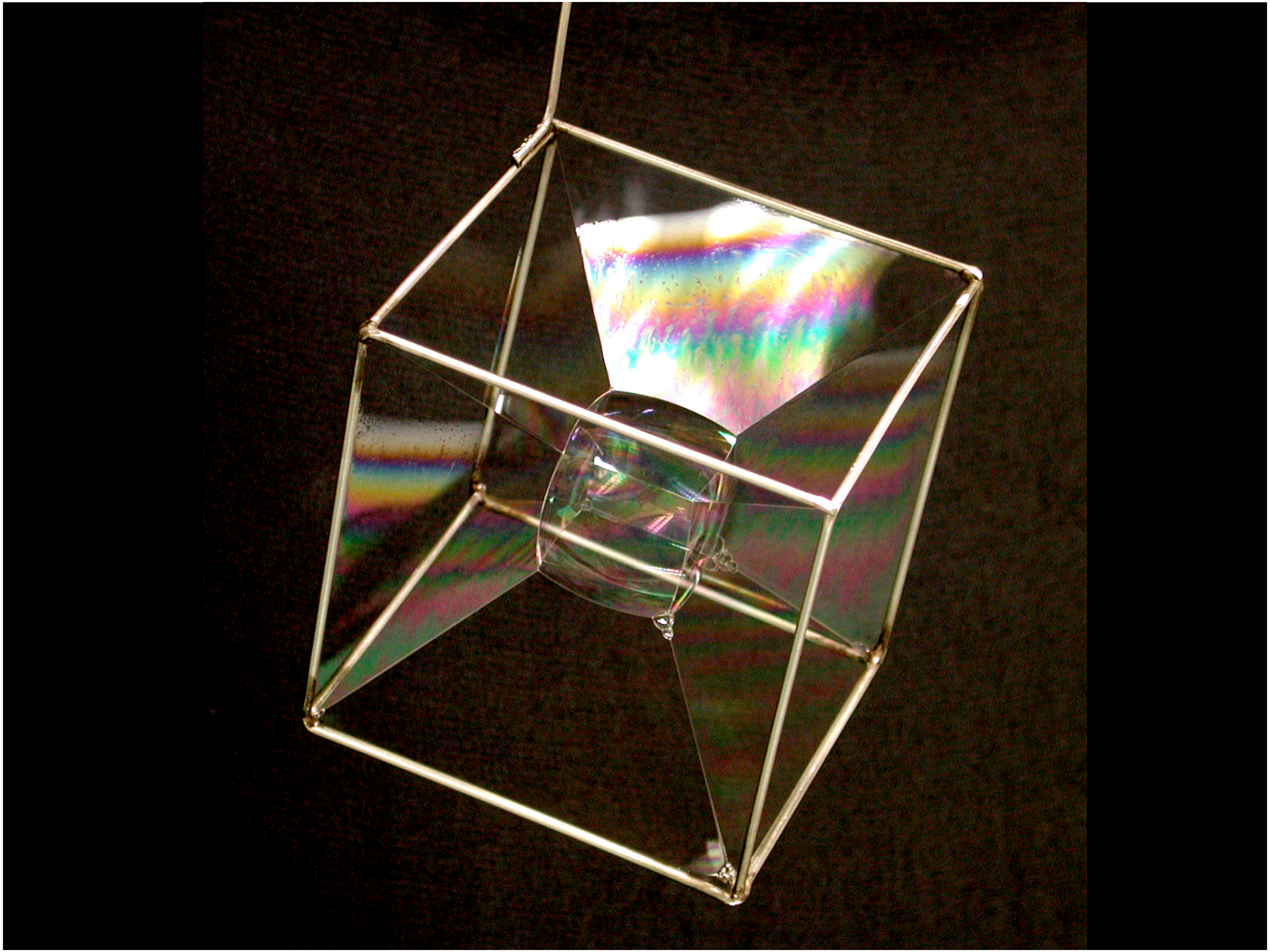
2. Minimal cones in $\mathbb{R}^3$.

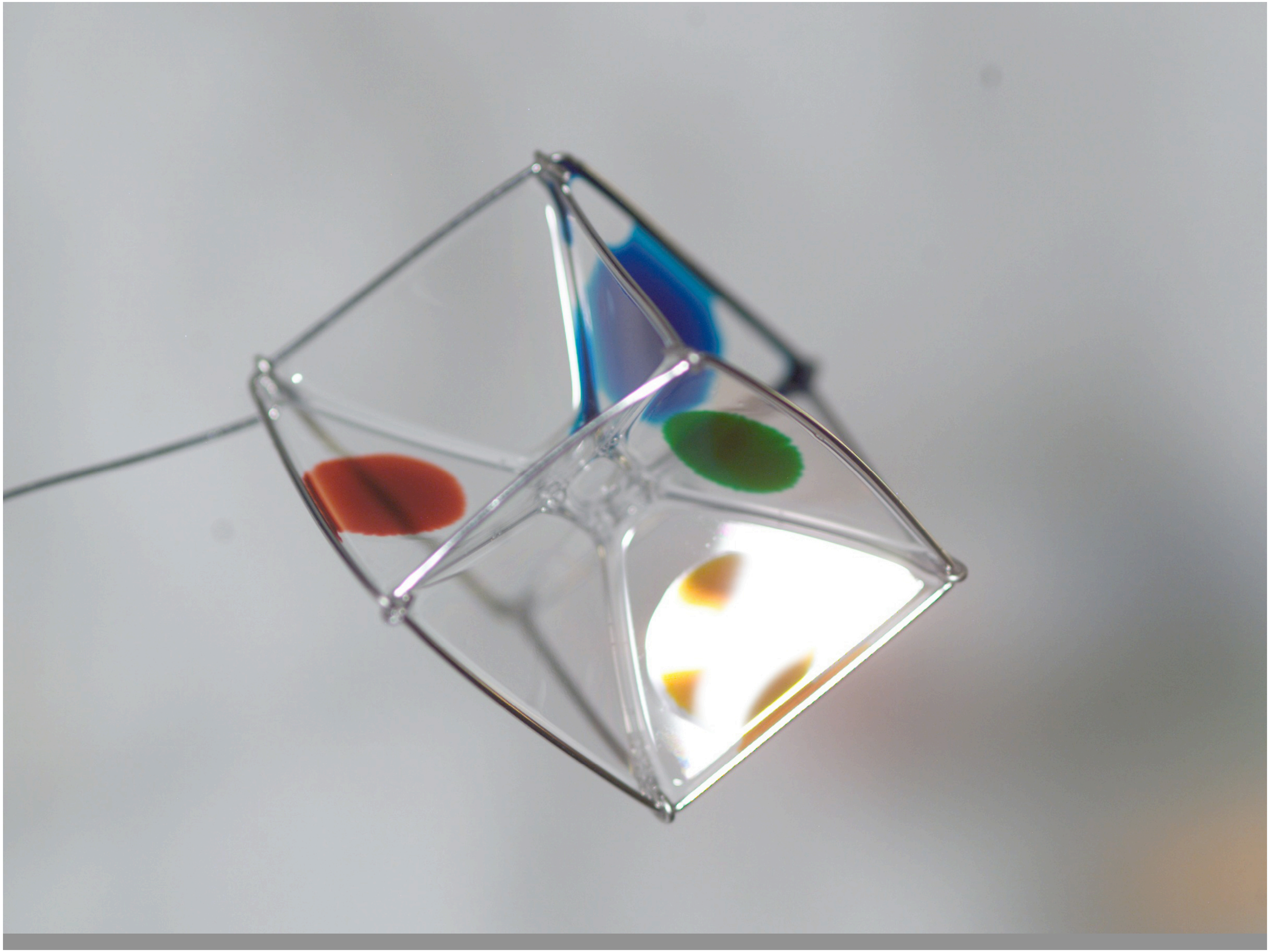
For $d = 1$, the cones are the lines and the Y (3 half lines with the same origin, and that make 120 degree angles). Even true in $\mathbb{R}^n$.

Locally every almost minimal set of dimension 1 looks like a line or a Y (modulo a C^1 diffeomorphism).

For $d = 2$ and $n = 3$, the minimal cones are the planes, the sets $\mathbb{Y} = Y \times \mathbb{R}$ (three half planes with 120 degree angles), and the sets $\mathbb{T}$ (cone over the union of the edges of a regular tetrahedron; they have 6 faces and 4 edges). Pictures.

Theorem [Jean Taylor, 1978]. *Locally, every almost-minimal set is C^1-equivalent to a minimal cone (as above) if h is small enough near* 0.



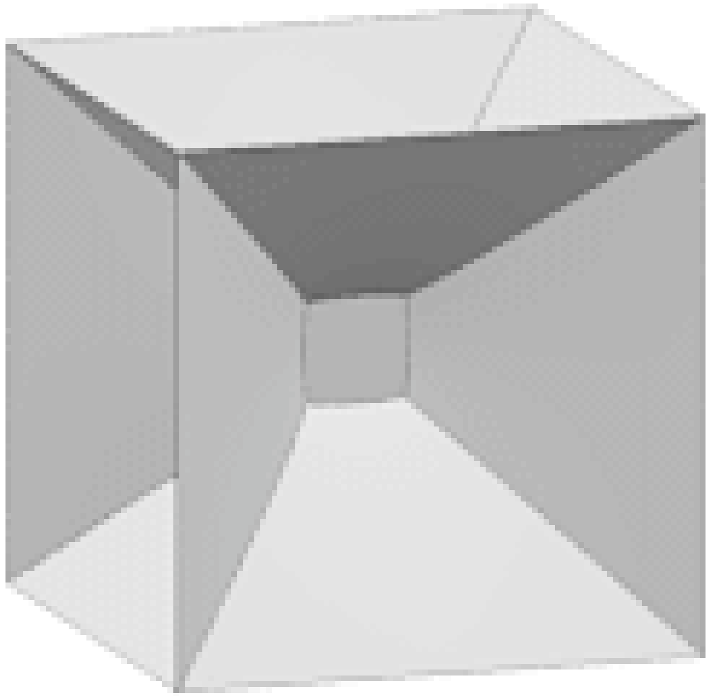


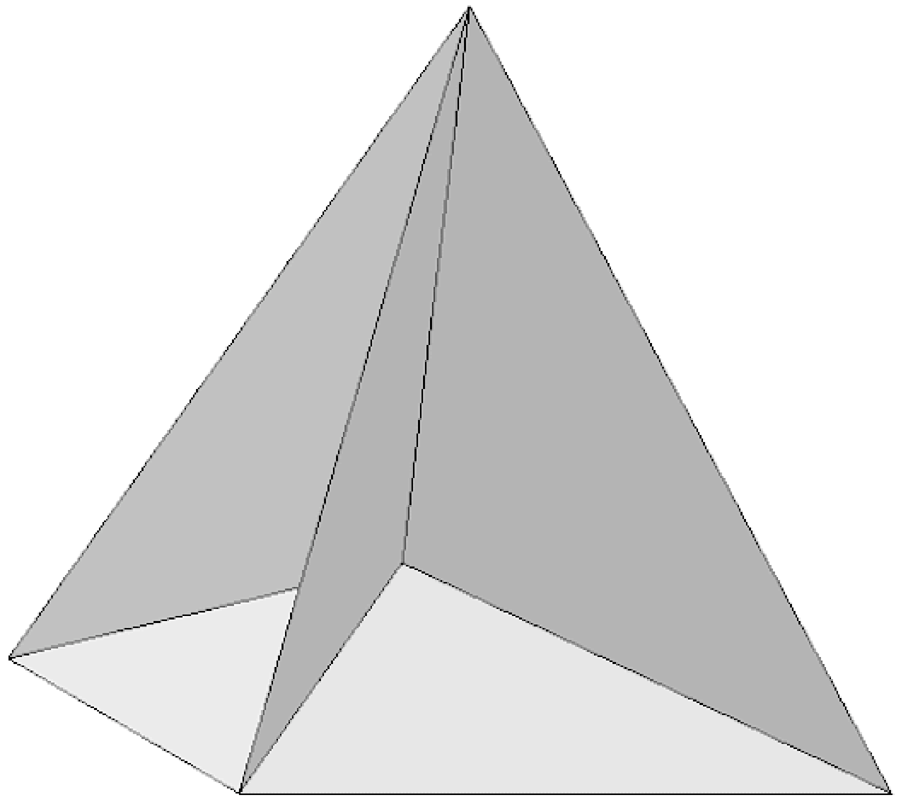
3. Minimal cones of dimension 2 in $\mathbb{R}^n$.

We have a (too) general description. Let E be such a cone. Set $K = E \cap \partial B(0,1)$.

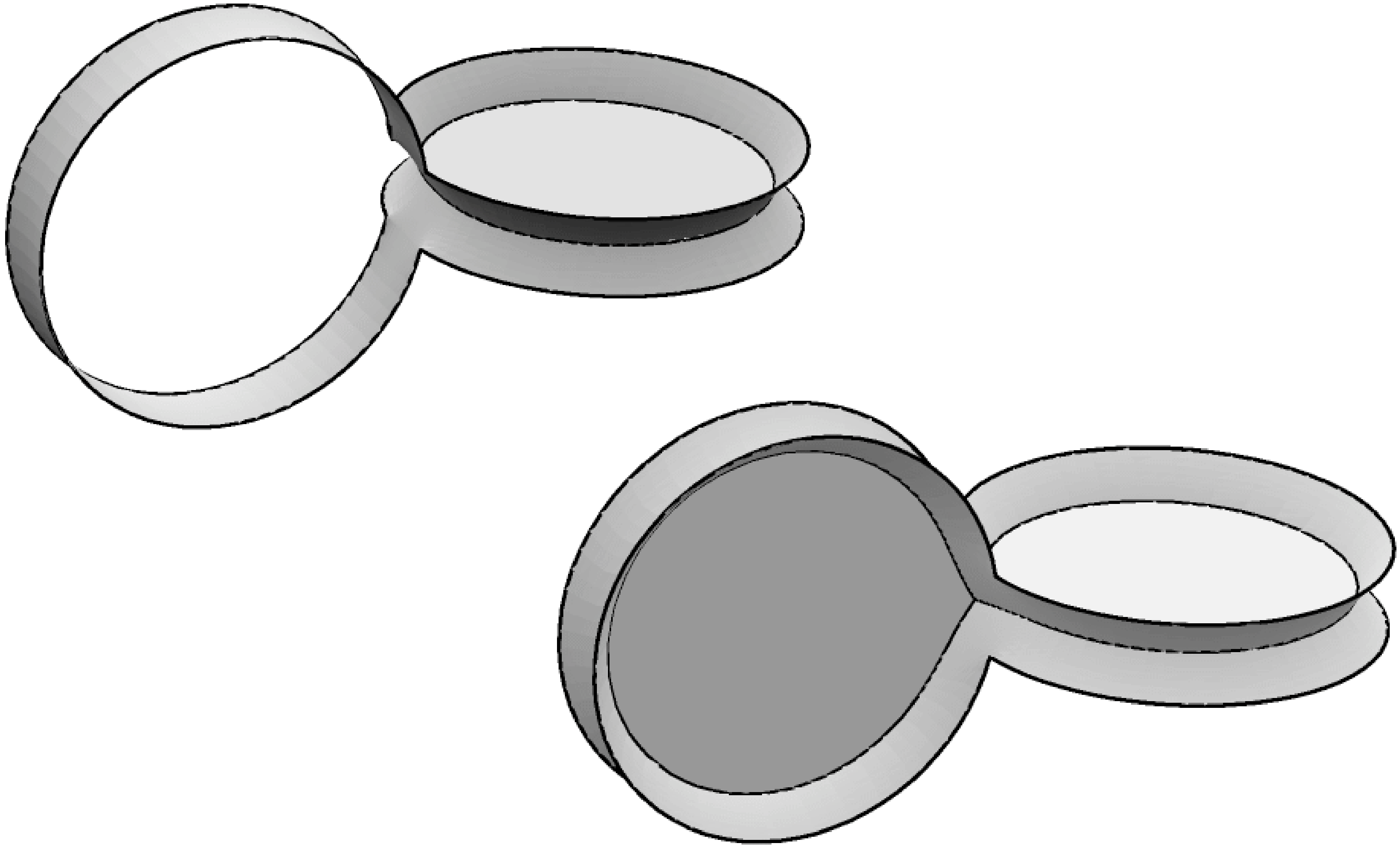
Then K is a finite union of circles and arcs of circles. The circles are far from the rest of K. At their ends, the arcs meet by sets of 3, with 120° angles (no free ends). The arcs are not too short.

Examples in $\mathbb{R}^3$:
A plane corresponds to a circle
$\mathbb{Y} = Y \times \mathbb{R}$ corresponds to three half circles meeting at the two poles
$\mathbb{T}$ comes from 6 arcs of circles (the projections of the edges of the trahedron).





Pictures from K. Brakke’s site



Pictures from K. Brakke’s site

More examples in $\mathbb{R}^4$:

Two disjoints circles gives a transverse union $P_1 \cup P_2$ of 2-planes. But is it minimal?

$Y \times Y$ (the product of two sets Y contained in orthogonal planes) corresponds to a net of 9 arcs of circles.
But is it minimal?

Is there a one-parameter family of minimal cones with K connected?

Incidentally: is every minimal 2-set in $\mathbb{R}^3$ (or $\mathbb{R}^4$) automatically a cone?

4. Local regularity of almost-minimal 2-sets in $\mathbb{R}^n$?

(besides Jean Taylor's theorem)

Let E be an almost-minimal 2-set in $\mathbb{R}^n$, $n \geq 4$, and let $x \in E$.

Known : E is, in some $B(x, r)$, biHölder-equivalent to a minimal cone.

But we don't have a list of minimal cones.

We can get the C^1-equivalence in some cases only, depending on the "full length property" of one (or all) tangent minimal cone(s) to E at x.

[Property concerning the variations of length for perturbations of K into other nets of geodesic arcs. We don't have counterexamples either.]

5. When is $P_1 \cup P_2 \subset \mathbb{R}^4$ minimal?

Théorème 1. *The union $P_1 \cup P_2$ of two orthogonal planes is minimal.*

This is classical, and relies on the following facts.

• Denote by π_j the orthogonal projection onto P_j. Then $\pi_j(F)$ contains P_j whenever F is a competitor for $P_1 \cup P_2$.

• If F is rectifiable if ds denotes a surface element of F, then

$$\pi_1(ds) + \pi_2(ds) \leq ds. \tag{6}$$

Amusingly false in dimension $d = 1$.
By the way, $L_1 \cup L_2$ is never minimal.

Proof. We shall use this like a calibration.
Out of some cube Q, $F = P_1 \cup P_2$. And on Q,

$$\begin{aligned}
\mathcal{H}^2(F \cap Q) = \int_{F\cap Q} ds &\geq \int_{F\cap Q} \pi_1(ds) + \pi_2(ds) \\
&\geq \mathcal{H}^2(\pi_1(F \cap Q)) + \mathcal{H}^2(\pi_2(F \cap Q)) \\
&\geq \mathcal{H}^2(P_1 \cap Q) + \mathcal{H}^2(P_2 \cap Q) \\
&= \mathcal{H}^2((P_1 \cup P_2) \cap Q). \qquad \square
\end{aligned}$$

Lemma. *If $P_1 \perp P_2$, $P_1 \cup P_2$ is the only mininimal set in Q s.t. $\mathcal{H}^2(E) \leq \mathcal{H}^2((P_1 \cup P_2) \cap Q)$ and $\pi_j(E) \supset P_j \cap Q$ for $j = 1, 2$.*

Proof. We check the equality cases in (6), and the minimality finally allows us to eliminate the remaining cases.

When is $P_1 \cup P_2$ minimal?

When they make small angles, we can pinch in the middle and $P_1 \cup P_2$ is not minimal.

Frank Morgan gives a conjectural condition on the angles, under which $P_1 \cup P_2$ should be minimal, and Gary Lawler shows that one can pinch when it is not satisfied. Partial converse below.

We focus on the almost orthogonal union $P^\varepsilon = P_1^\varepsilon \cup P_2^\varepsilon$, where

$$|\langle v_1, v_2 \rangle| \leq \varepsilon |v_1||v_2| \text{ for } v_1 \in P_1^\varepsilon \text{ and } v_2 \in P_2^\varepsilon.$$

Theorem (Xiangyu Liang). *If $\varepsilon > 0$ is small enough, P^ε is minimal.*

6. Scheme of a proof modulo Plateau

Recall $P^\varepsilon = P_1^\varepsilon \cup P_2^\varepsilon$. Suppose that, for a sequence of ε that tends to 0, P^ε is not minimal.

Let $E^\varepsilon = f(P^\varepsilon)$ be a better competitor in the unit cube Q. We look for a contradiction. Unfortunately, no known algebraic trick as above.

Easy: f should not be injective. But we need to show that we save less by pinching than we lose by rotating the P_j before.

Note that for ε small, (6) almost holds and pinching pays very little.

Things would be easier if E^ε minimized $\mathcal{H}^2(E \cap Q)$ among deformations of P^ε in Q. So that we can use the minimality of E^ε in Q. Unfortunately, no known result seems to give such an E^ε. Bul let us pretend anyway (a more complicated fix exists).

Denote by π_1 and π_2 the orthogonal projections on P_1^ε and P_2^ε.
We may assume that $P_1^\varepsilon = P_1$.
We take Q with faces parallel to the P_j.

Take a sequence of ε that tends to 0 such that E^ε tends to a limit E^∞. Each E^ε is minimal inside Q, so (by a theorem on limits) E^∞ is minimal inside Q.

Next $\pi_j(E^\varepsilon) \supset P_j^\varepsilon$ (because E^ε is a deformation of P^ε), hence $\pi_j(E^\infty) \supset P_j$ (take limits).

Also $\mathcal{H}^2(E^\varepsilon \cap Q) < \mathcal{H}^d(P^\varepsilon \cap Q)$ by definition of E^ε, hence $\mathcal{H}^2(E^\infty \cap Q) \leq \mathcal{H}^d(P^\varepsilon \cap Q)$ by a theorem on the lower semi-continuity of $\mathcal{H}^d$ along sequences of quasiminimal sets.

The lemma says that $E^\infty = P_1 \cup P_2$. That is,

$$(*) \qquad E^\varepsilon \text{ tends to } P_1 \cup P_2.$$

Let $\delta > 0$ be small.

We want to find an origin x_0 and a radius r_0 such that E^ε is δr_0-close to $x_0 + P_1 \cup P_2$ in $B(x_0, r_0)$ but $r_0/2$ does not work.

At large scales, E^ε looks a lot like $P_1 \cup P_2$, so (for ε small) $x_0 = 0$ and $10^{-2} \leq r_0 \leq 1$ would work.

When (x, r) works, we try to find $(x', r/2)$. We stop when we cannot find x' any more. If we never stop, easier argument.

By construction, E^ε is $20r\delta$-close to an $x + P_1 \cup P_2$ in every $B(x_0, 10r) \setminus B(x_0, r)$, $r \geq r_0$.

By Jean Taylor' theorem and gluing, E^ε is composed of two nice C^1 graphs out of $B(x_0, 2r_0)$, E_1^ε (horizontal) and E_2^ε (vertical).

Define cylinders $V_j(r) = (\pi_j^\varepsilon)^{-1}(B(x_0, r))$ for $r > r_0$, $j = 1, 2$.

First cut E^ε in three: choose $r \in (2r_0, 4r_0)$, and write $E^\varepsilon = F \cup F_1 \cup F_2$, with $F = E^\varepsilon \cap V_1(r) \cap V_2(r)$, $F_1 = E_1^\varepsilon \setminus V_1(r)$, and $F_2 = E_2^\varepsilon \setminus V_2(r)$. First ty to estimate brutally:

$$\begin{aligned} \mathcal{H}^2(E^\varepsilon) &= \mathcal{H}^2(F) + \mathcal{H}^2(F_1) + \mathcal{H}^2(F_2) \\ &\geq \mathcal{H}^2(F) + \mathcal{H}^2(\pi_1^\varepsilon(F_1)) + \mathcal{H}^2(\pi_2^\varepsilon(F_2)). \end{aligned} \tag{1}$$

For $j = 1, 2$, $\mathcal{H}^2(P_j^\varepsilon) = \mathcal{H}^2(\pi_j^\varepsilon(F_j)) + \mathcal{H}^2(\pi_j^\varepsilon(V_j(r)))$ (disjoint union). We subtract both things from (1) and get that $\mathcal{H}^2(E^\varepsilon) - \mathcal{H}^2(P_1^\varepsilon \cup P_2^\varepsilon) \geq \mathcal{H}^2(F) - \mathcal{H}^2(\pi_1^\varepsilon(V_1(r))) - \mathcal{H}^2(\pi_2^\varepsilon(V_2(r)))$ a contradiction with the definition of E^ε if we show that

$$\mathcal{H}^2(F) \geq \mathcal{H}^2(\pi_1^\varepsilon(V_1(r))) + \mathcal{H}^2(\pi_2^\varepsilon(V_2(r))). \tag{2}$$

Recall that we would like

$$\mathcal{H}^2(F) \geq \mathcal{H}^2(\pi_1^\varepsilon(V_1(r))) + \mathcal{H}^2(\pi_2^\varepsilon(V_2(r))). \tag{2}$$

But now the analogue of (6) on page 9 is that

$$\pi_1(ds) + \pi_2(ds) \leq (1 - C\varepsilon)ds \tag{3}$$

(for surface elements ds in F), which merely yields

$$(1 - C\varepsilon)\mathcal{H}^2(F) \geq \mathcal{H}^2(\pi_1^\varepsilon(V_1(r))) + \mathcal{H}^2(\pi_2^\varepsilon(V_2(r))). \tag{4}$$

So we shall get a contradiction if we can improve the estimates above by more than $C'\varepsilon r_0^2 \geq C\varepsilon\mathcal{H}^2(F)$.

Recall that for $2r_0 < r < 3r_0$, $E^1_\varepsilon \cap \partial V_1(r)$ is the graph over the circle $c(r) = P_1^\varepsilon \cap \partial V_1(r)$ of a nice C^1 function f.

Case 1. We can find $r \in (2r_0, 3r_0)$ such that

$$\int_{c(r)} |f(x) - m_{c(r)} f|^2 dx \geq \delta_1^2 r^3 \tag{5}$$

for some small $\delta_1 << \delta$) to be chosen later; $m_{c(r)} f$ is the mean value. We know that F_1 is the graph over $P_1^\varepsilon \setminus V_1(r)$ of a nice C^1 function g, with $g = f$ on the boundary. Standard estimates on harmonic functions yield $\int |\nabla g|^2 \geq c\delta_1^2 r^2$, and then

$$\mathcal{H}^2(F_1) \geq \mathcal{H}^2(P_1^\varepsilon \setminus V_1(r)) + c\delta_1^2 r^2 \tag{6}$$

which is more than enough (if ε is small).

Case 2. No r can be found as above, nor with respect to P_2^ε.

Recall that by minimality of r_0, E^ε is $\delta r_0/2$-far from all $x+P_1\cup P_2$ in $B(x_0,r_0)$.

By a compactness argument, E^ε is also $\delta_2 r_0$-far from all $x+P_1\cup P_2$ in $V_1(3r_0)\cap V_2(3r_0)\setminus V_1(2r_0)\cap V_2(2r_0)$. Here $\delta_2>0$ is very small, depending on δ.

Then E_1^ε is $\delta_2 r_0$-far from all planes in $V_1(3r_0)\setminus V_1(2r_0)$ (or the same thing with the vertical part).

Take $\delta_1 << \delta_2$. By definition of Case 2, every $E_1^\varepsilon\cap\partial V_1(r)$, $2r_0<r<3r_0$ is very close to a circle.

Then two of these circles (say with $2r_0<r<r_1<3r_0$) are at different altitudes (more than $\delta_2 r_0/2$).

We further cut F_1 into $F_{1,1} = E_1^\varepsilon \cap V_1(r_1) \setminus V_1(r)$ and $F_{1,2} = E_1^\varepsilon \setminus V_1(r_1)$ and say that

$$\mathcal{H}^2(F_1) = \mathcal{H}^2(F_{1,1}) + \mathcal{H}^2(F_{1,2}) \geq \mathcal{H}^2(F_{1,1}) + \mathcal{H}^2(P_1^\varepsilon \setminus V_1(r_1))$$

and

$$\mathcal{H}^2(F_{1,1}) \geq \mathcal{H}^2(P_1^\varepsilon \cap V_1(r_1) \setminus V_1(r)) + c\delta_2^2 r_0^2$$

because of the different (almost circular) boundary values and a simple estimate on gradients. We add and get that

$$\mathcal{H}^2(F_1) \geq \mathcal{H}^2(P_1^\varepsilon \setminus V_1(r)) + c\delta_2^2 r_0^2,$$

a sufficient improvement. □

How to manage without Plateau?

The previous argument shows that some existence results for Plateau-like problems could be useful. But here we can manage without this.

There is an argument by V. Feuvrier that constructs a minimal set E^ε in Q, starting from a correctly modified minimizing sequence $\{E_k^\varepsilon\}$ of deformations of $P_1^\varepsilon \cup P_2^\varepsilon$ in Q.

Now E^ε is not necessarily a deformation of the E_k^ε. But its projections still contain the P_j^ε, and eventually we can apply the uniqueness result above to show that $E^\infty = P_1 \cup P_2$.

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