Mathematics and Finance: End of the Bubble?

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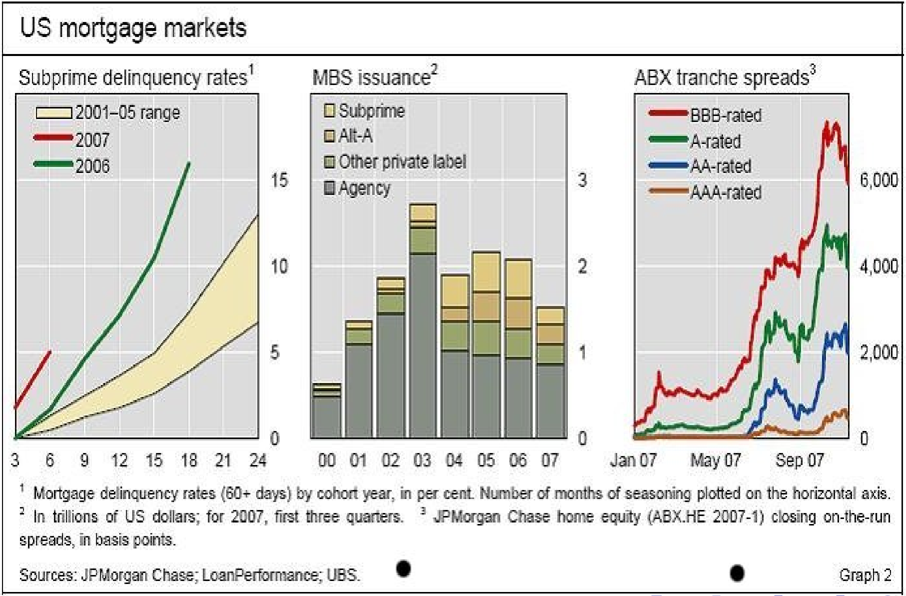
Plan

Subprime crisis I

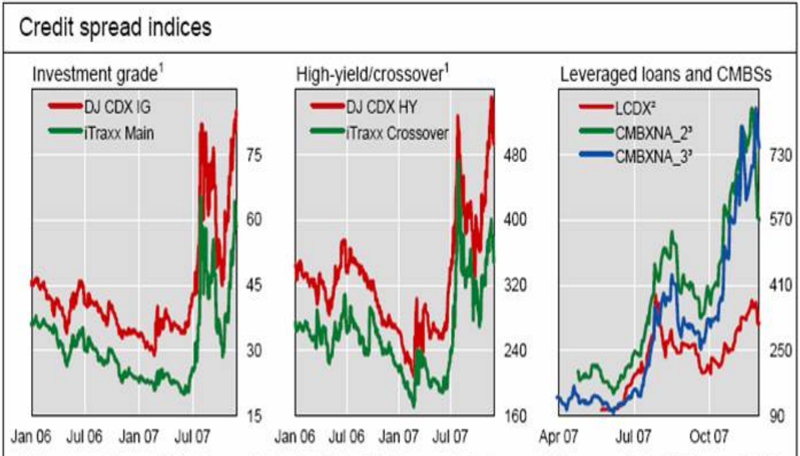
- the excesses of the finance industry are dragging down the whole economy.
- Credit crunch was based on subprime risks, a lowering of underwriting standards that drew people into mortgages.
- Diffusion of the home mortgage crisis in any financial places through securization via MBS
- Mortgage-backed securities (MBS) depend of the performance of hundreds of mortgages.

Subprime delinquency rates

400 billions dollars in 2004 to 1 400 billions in 2007



Credit Spreads Indices



[1] Five-year on-the-run CDS mid-spread, in basis points. [2] Index spreads for five-year CDSs on syndicated US loans. [3] Index spreads for BBB tranches of commercial mortgage-backed securities (index series 2 and 3).

Source: JPMorgan Chase.

Delinquency practices I

- Lapses in the evaluation of MBS and their derivatives by rating agencies and investor
- Many of these securities received a “good”rating, and their returns were significantly greater then comparable rated bond
- The practice of “rating arbitrage” getting better-than merited rating and selling securities based on that rating was born.
- Investors in MBS were assured to have a AAA asset, and suddenly find that the MBS is a *junkbonds*
- In complete absence of prudential regulation and oversight (Mr.Greenspan)

Don't Blame the Quants I

said Steve Shreve, in (Forbes.com, the 10.08.08)

- S.Shreve is Maths Professor at Carnegie Mellon University, and responsable of a Master Program in Quantitative Finance.
- Students become Quants= Quantitative people (PhD's in Math or Physic) in Investment Banks.

The Blames

- People argue that without quants models, these complicated MBS might not have been created.
- It is only partially true, since it arrives that financial products are sold, before to have a good pricing model

Don't Blame the Quants II

- Even if the growth of credit derivatives market was exceptional, it was not the same for the quantitative research in the aera
- Nevertheless, quants have produced models to price such derivatives, (often with limitations on their use),
- and they have to assume this responsibility in the crisis
- even their are not decision- makers
- even their are not designer of derivatives securities

Quantitative Finance : Three Pillars I

Practice

- Financial innovation
- Pricing
- Risk management

Mathematics

- Continuous Time Finance
- Stochastic Calculus
- Risk measure

Numerical implementation

- Modelling
- Calibration
- Risk management in Practice

An Historical Example : paper currency I

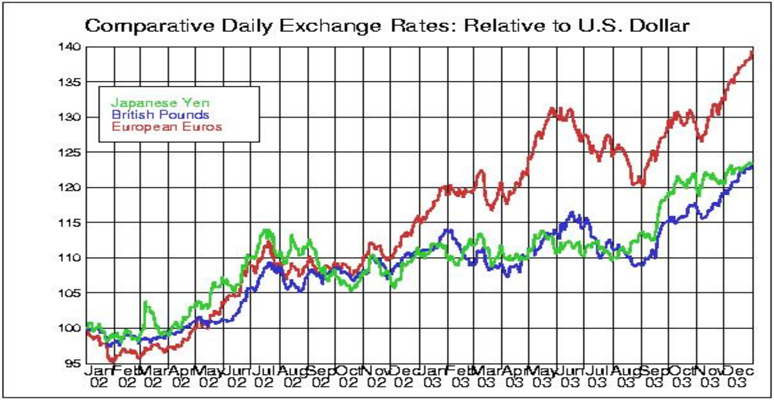
from William Poole, (Pres.Fed Saint Louis)

- the starting point was goldsmith receipts accepted as a medium of the exchange
- benefit : economy on the use of gold, and encourage economic activity and trade
- source of instability : when some bankers issued too many notes against the gold they had.
- panic and lose of confidence
- inflation When governments issued a lot of currency.

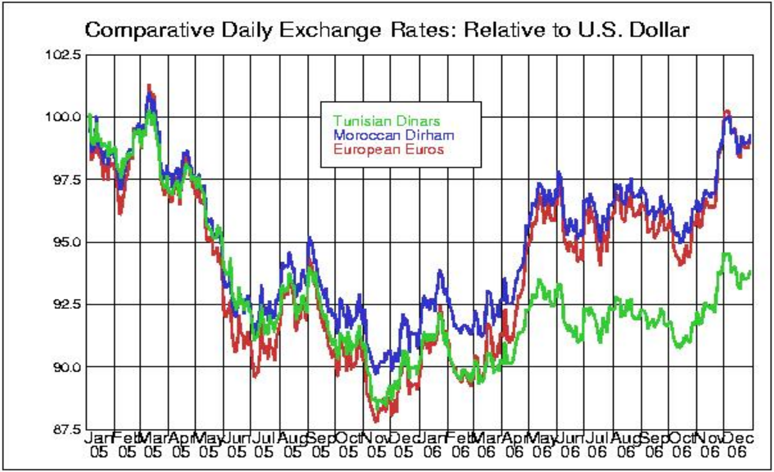
Since 1914, Governments have never abandoned paper currency, and try to control inflation, by mandating price stability as an objective for monetary policy

Deregulation and Fluctuations

Deregulation (1970) would not have been possible without helping economic agents to manage their financial risks.



Dinars/US, Year 2006

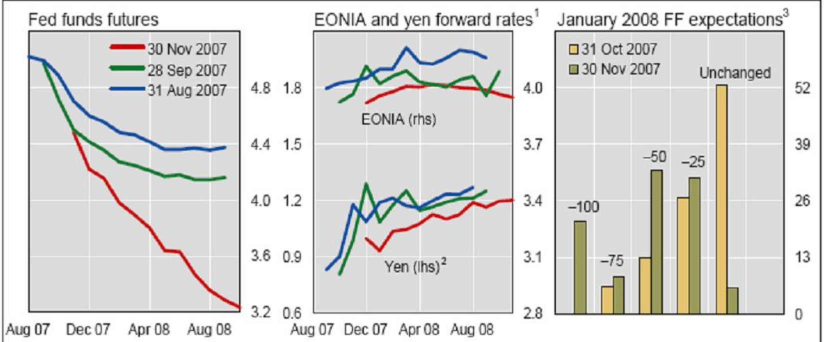


Forward contrat I

Financial Innovation : by allowing economic agents to made financial operations in the future at today fixed price,

- forward and futur contract, that obligated one counterpart to buy and the other to sell a fixe amount of securities at an agreed date in the future T.
- future contract are the standardized version of forward contract by clearinghouses, or new market
- used as a protection again large movement of the market
- swap contracts are some extension of forward contract to a series of cash flows at specified date in the future (interest-rates, currency)
- Based on new technology, computer

IRS Forward 2007-2008



[1] Implied one-month rates. [2] Calculated from Libor fixing rates. [3] Probabilities of changes in the federal funds target rate (FF) implied by prices of fed funds futures and options. The values above the bars represent total reductions in FF between the dates indicated and end-January 2008 (in basis points), compared to the prevailing level of 4.5%. The probabilities on the vertical axis are expressed in per cent.

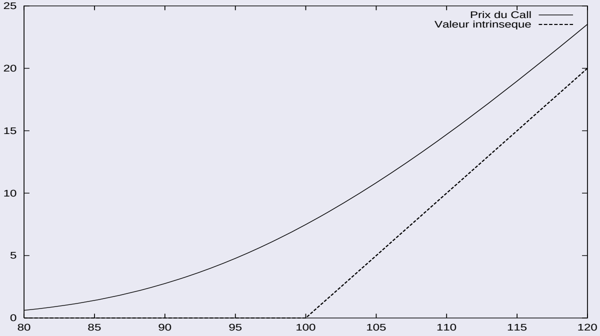
Sources: Bloomberg; BIS calculations.

Graph 9

Definition (Options Contract)

Call (Put) Options are simply

- the right, but not the obligation,
- to buy (sell) something in the future
- at given price called = exercise price = strike price= K



Definition

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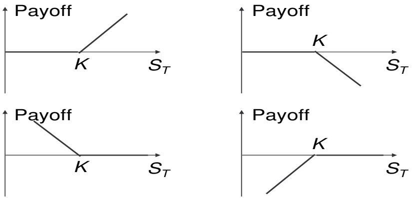
Les options d'achat et de vente

Les payoffs (ou valeur à maturité en T) sont des options en fonction du prix du sous-jacent ***ST***



Market Risk Industry

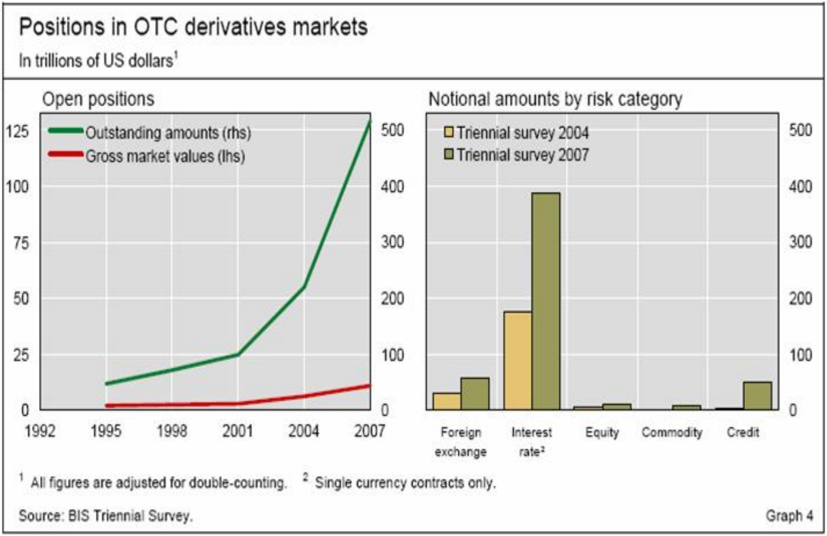
More than $20 trillions in notional each year

- contracts (futures, options, swaps), or more complex options : Barrier options, Asian options, American options .
- Various underlying : stocks, currencies, interest rates, commodities... called basic securities

New non financial supports

- Credit derivatives
- New markets and their derivatives :
  catastrophe bonds, energy and weather derivatives, CO_2-market.

OTC Derivatives 99-2007, Bis Report



Pricing and Hedging Problem

- Pricing Problem

  What is the price of these contracts ? It depends on the (uncertain) future values of underlying.

- Hedging Problem

  From the opening in 1973 of the first derivatives market in Chicago (CBOT)

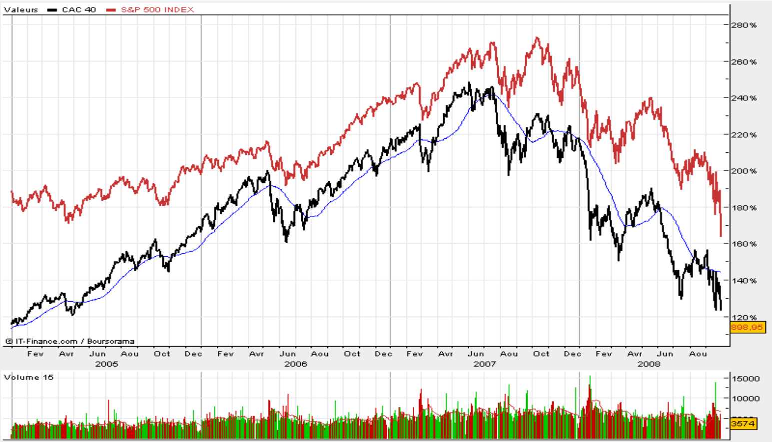
  How to reduce the exposure of the option's seller ?

- Different risk for buyer or seller
  - The buyer exposure is the premium = option price
  - The seller exposure may be very large $(X_T - K)^+$

Both problems have to be considered together.

Trend and Price fluctuations

CAC40/SP500,2005-2008



European Indices

CAC40+FTSE, 96-08

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Louis Bachelier I

Surprisingly, the starting point of financial risk industry expansion is

- Brownian motion theory (Einstein (1905), Wiener(1913), Levy (1930), Itô (1940..))
- and Itô's stochastic calculus

first introduced in 1900 in his PhD's Thesis "Théorie de la Spéculation" by the French Mathematician

The Mathematician Louis Bachelier

before

- Paul Appel, President
- Henri Poincaré, Examinator and Director
- Joseph Boussinesq, Examinator

Louis Bachelier Jeune

The Black & Scholes Paradigm of Zero-Risk

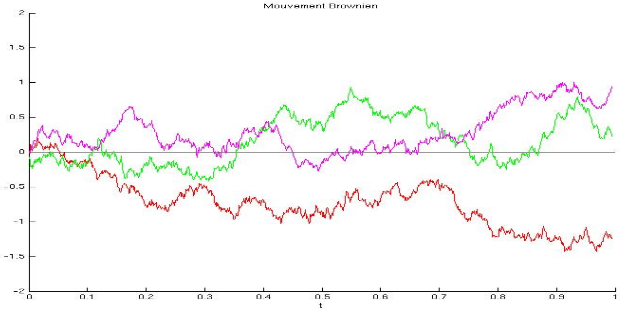
I

- Black, Scholes and Merton showed in 1973 that the option's seller may deliver the contract at maturity without incurring any residual risk, by using a dynamic trading strategy.
- Forward contract can be replicated by static strategy, without any model
- The B&S formula (giving the price and the hedge of Call options) was implemented in the CBOT a few months later.
- This totally new economic idea was rewarded by the Nobel price (1997).
- But, it did not prevent Long Term Capital Market's from Bankruptcy(1998)(Another story).

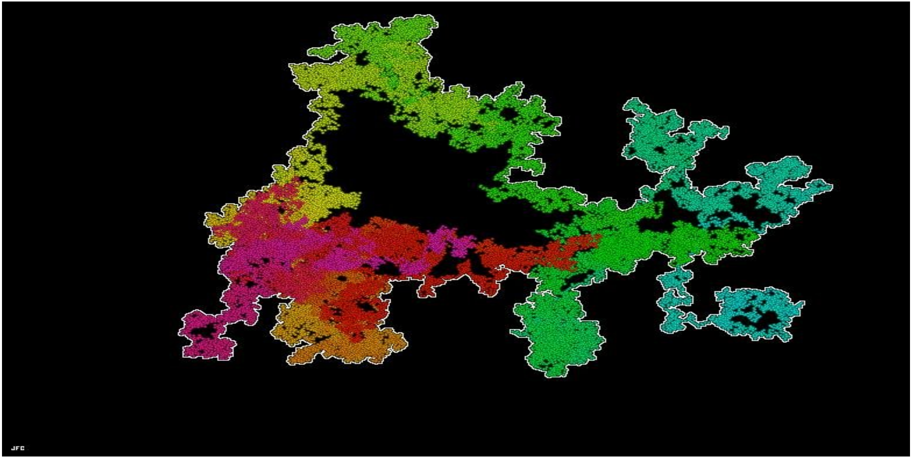
A Dynamic Uncertain World

• Uncertainty is modelled via a family Ω of scenarios ω, the possible (continuous) trajectories of the asset prices, $X_t(\omega) = \omega(t)$.

Example, Brownian motion paths



2D Brownian motion



Thanks to J.F Colonna (Ecole Polytechnique)

Stochastic Calculus

- By assumption, the paths have a finite continuous quadratic variation : (D_n)=the sequence of dyadic partitions).

$$[X]_t(\omega) = \lim_n \sum_{t_i \le t, t_i \in D_n} (X_{t_{i+1}} - X_{t_i})^2$$

Itô's formula

$$f(t, X_t)(\omega) = f(0, x_0) + \int_0^t f'_x(s, X_s)(\omega)\, dX_s(\omega)$$

$$+ \int_0^t f'_t(s, X_s)(\omega)\, dt + \int_0^t \frac{1}{2} f''_{xx}(s, X_s)(\omega) d[X]_s(\omega)$$

The last integrals are well defined as Lebesgue-Stieljes integrals.

The first integral exists as Itô's integral, defined as limit of non-anticipating Rieman sums, (where we put $\delta_t = F'_x(t, X_t)$),

$$\sum_{t_i \leq t, t_i \in D_n} \delta_{t_i}(\omega)(X_{t_{i+1}} - X_{t_i})(\omega).$$

From a financial point of view, the Itô's integral is the cumulative gain process of trading strategies :

- δ_t is the number of the shares held at time t
- the increment in the Rieman sum is the price variation over the period.
- the non-anticipating assumption corresponds to the financial requirement that the investment decisions are based only on the past prices observations.

Self-financing Portfolios I

The residual wealth of the trader is invested only in cash, with a yield rate (called short rate) r_t by time unit.
The self-financing condition states that the wealth increment is only generated by : - the gain due to the risky investment $\delta_t dX_t$
- the interest due to the residual wealth $V_t - \delta_t X_t$

$$dV_t = r_t(V_t - \delta_t X_t)dt + \delta_t dX_t = r_t V_t dt + \delta_t(dX_t - r_t X_t\, dt), \; V_0 = z$$

About the information structure

In the B&S framework, the strategy δ_t is only based on today's prices $\delta_t = \delta(t, X_t)$. This makes reference to the hypothesis of market efficiency : all available market information is reflected in today's prices.

Hedging Derivatives : a Solvable Target Problem I

- Let us consider a trader having to pay the pay-off $\phi(X_T)(\omega)$ in the scenario ω, ($(X_T(\omega) - K)^+$ for a Call option) at time T.
- This target may be hedged(approached) in all scenarios by the wealth generated by a self-financing portfolio, solving

Backward Stochastic Differential Equation

$$dV_t = r_t(V_t - \delta_t X_t)dt + \delta_t dX_t = r_t V_t dt + \delta_t(dX_t - r_t X_t\, dt),\ V_T = \phi(X_T)$$

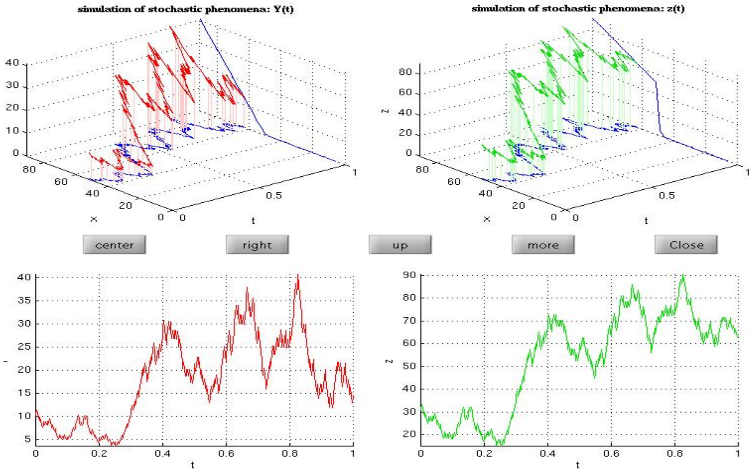
- The "miraculous" message in the world of Black and Scholes is that a perfect hedge is possible and easily computable.

Hedging Derivatives : a Solvable Target Problem II

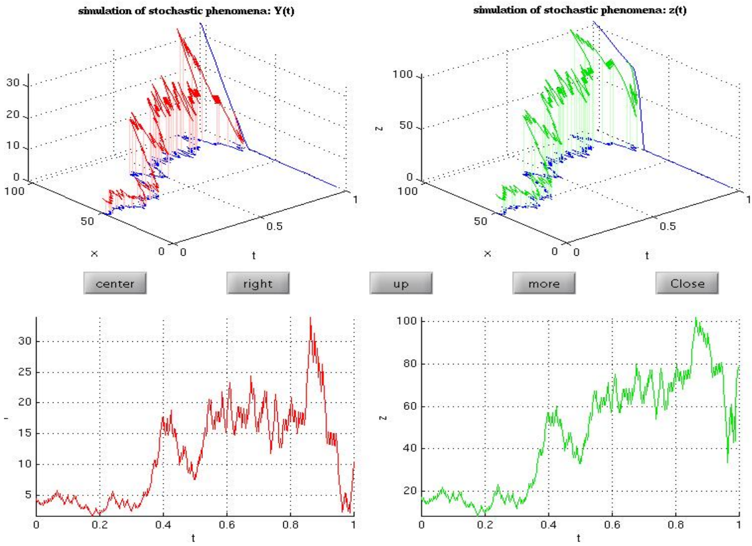
- Pricing rule *The price at time t of the derivative is the Value of the hedging portfolio :* otherwise it is possible to make profit without bearing any risk.
- Such strategy, an arbitrage, is prohibited in a liquid market. So, no Arbitrage implies Price Uniqueness

Call(50,50) : Hedging portfolio of Call

- blue= asset path
- red= portfolio value, green= portfolio's risky part



Call(50,70) : Hedging portfolio of Call



Pricing Partial Differential Equation

Assume quadratic variation satisfying :

$$d[X]_t(\omega) = \sigma^2(t, X_t(\omega))\, X_t^2(\omega) dt.$$

The function $\sigma^2(t, X_t(\omega))$ is a key parameter callec local volatility.

Theorem (Pricing and hedgin)

Let u be a regular solution of Pricing PDE

$$u'_t(t,x) + \frac{1}{2} u''_{xx}(t,x) x^2 \sigma^2(t,x) + u'_x(t,x) x\, r - u(t,x)\, r = 0\,, u(T,x) =$$

$u(t, X_t)$ is the option price at time t and the hedging portfolio is given by

$$\delta(t, X_t) = u'_x(t, X_t)$$

Pricing Kernel

- Let $q(t,x,T,y)$ be the Pricing PDE's fundamental solution
- The pricing rule becomes : $u(t,x) = \int h(y)q(t,x,T,y)dy$. q is also called *pricing kernel*.

Risk-neutral Pricing

With some additional assumptions, there exists a $\mathbb{P}$ equivalent probability measure $\mathbb{Q}$, called the risk neutral probability s.t.

$$u(t,x,T) = \mathbb{E}_{\mathbb{Q}}\Big[h(X_T)|X_t = x\Big]$$

- The function $\sigma(t,x) = \sigma_t$ is called the volatility function function,
- Useful representation for Monte Carlo simulation

Black and Scholes Formula

For deterministic volatility

- There exists a closed formula for Call option prices, the famous B&S Formula.

$$C_{BS}(t,x,K,T,\sigma) = x\,N(d_1) - Ke^{-r(T-t)}\,N(d_0)$$

$$d_0 = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\ln[\frac{x}{Ke^{-r(T-t)}}] - frac12\sigma\sqrt{T-t}$$

$$d_1 = d_0 + \sigma\sqrt{T-t}$$

where$N(.)$ is the cumulative function of the normal distribution.

- The hedging portfolio is based on $\Delta_t = N(d_1)$ risky assets.

Classical Framework and Market Trend

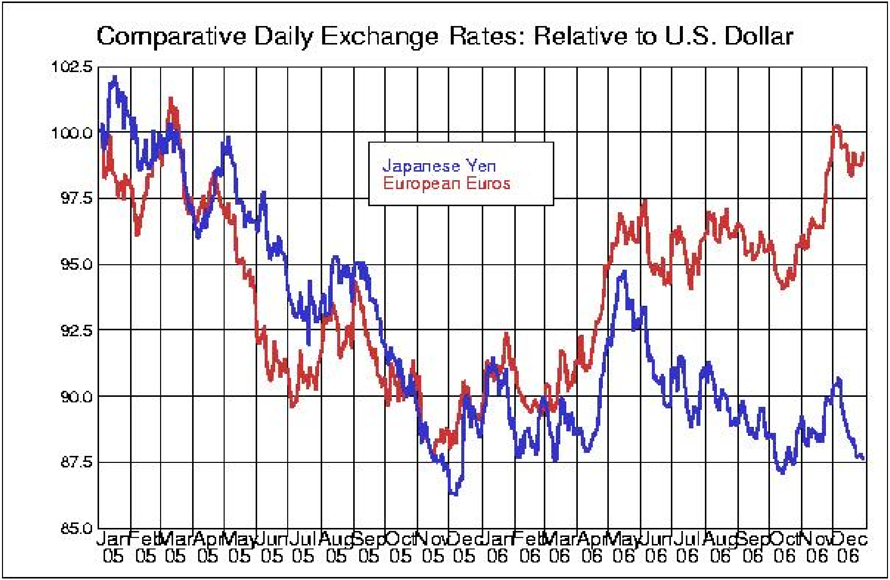
Following Bachelier, asset price dynamics are driven by a Brownian motion (BM) via Stochastic Differential Equation (SDE)

$$dX_t = X_t(\mu(t, X_t)dt + \sigma(t, X_t)dW_t), \;\; X_{t_0} = x_0$$

W may be viewed as a standardized Gaussian noise.

- The local expected return $\mu(t, X_t)$ is a trend parameter appearing for the first time in our propose.
  That is key point in financial risk management : Call prices do not depend on the market trend.
- It seems surprising, since the first motivation of this financial product is to hedge the purchaser against increases in the underlying prices.
- By using a dynamic hedging strategy, the trader (seller) is also protected against an unfavorable evolution.

Euro and Yen against Dollar



Options on New Underlying and Statistics I

The most reliable data are historical data. The different steps of the calibration are the following

- Modelling the underlying, for instance as Markov diffusion
- Using statistical procedures to estimate diffusion parameters, trend and volatility
- Solving the Pricing PDE to deduce price and hedge
- or Using Monte Carlo Methods

Options on New Underlying and Statistics II

- However, traders are hesitant to use historical estimators : they argue that financial markets are not “statistically” stationary and that the past is not sufficient to explain the future.
- In general, in this context it is not necessary to use sophisticated statistical methods because the spread bid-ask is very large.

Liquid Market and Implied Volatility I

Organized Markets : CBOT, NYSE, LIFE, MATIF, Currencies....

Implied Volatiliy

- When possible, traders use the additional information given by the quoted option prices
- translate it into volatility parametrization via the B&S formula.
- The *implied volatility*, Σ^{imp} is defined as :

$$C^{obs}(T,K) = C^{BS}(t_0, x_0, T, K, \Sigma^{imp})$$

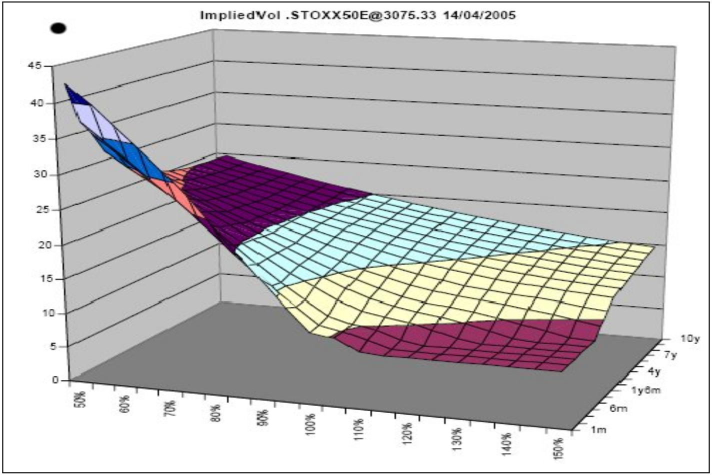
Implied Volatility (2)

Operational motivations

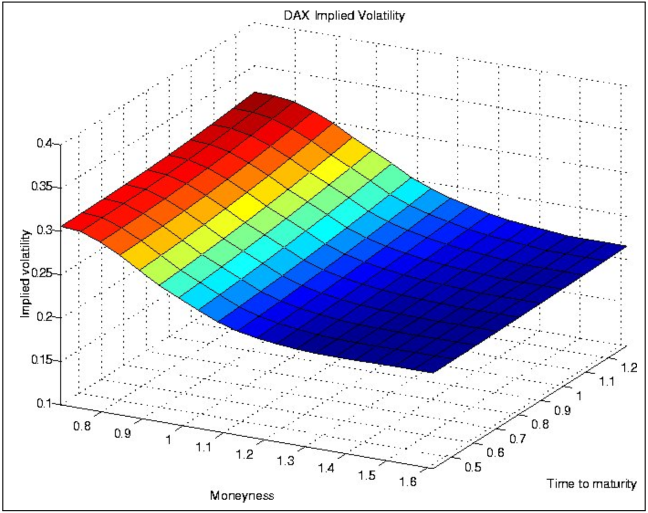
- Every day after the market closing, traders have to evaluate their portfolio with a model fitted by quoted prices .
- The option hedge portfolio is easily computed by $\Delta^{imp} = \partial_x C^{BS}(t_0, x_0, T, K, \Sigma^{imp})$.
- This strategy is used dynamically, by defining the implied volatility and the associated Δ at any renegotiation date.

Implied Volatility and Smile

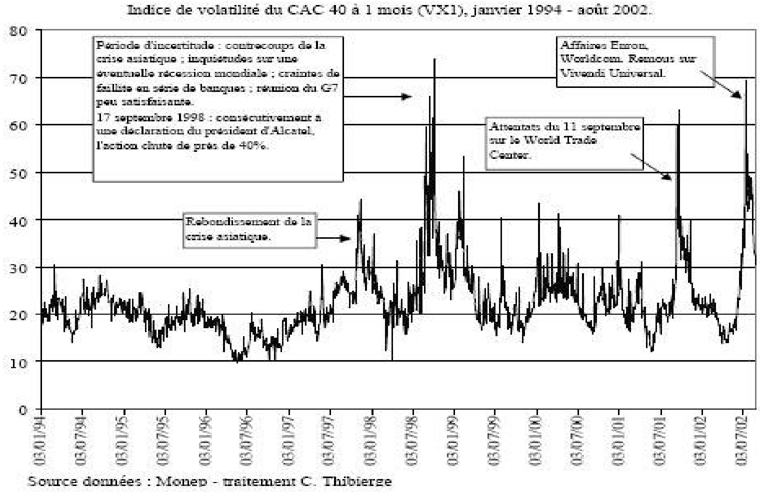
Implied Volatility Surface/ SP500



Average Volatility Surface/Dax



French Index Volatility



Implied Volatility Surface : An Ill-posed Inverse Problem

Unfortunately, the option market is typically limited to relatively few different options, and a naive interpolation yields to irregularity and instability of the local volatility .

Calibrating local volatility

- Finding the local volatility is a non-linear problem of function approximation from a finite data set.
- Given that data are solutions at time (t_0, x_0) of Pricing PDE, ill-posed inverse Problem tools provide robust solutions.
- Difficulties due to the non $\mathcal{C}^2$ regularity of the call function
- Managing Model Risk in place of Market risk

An example : Osher's optimization program

Osher (1996) looks for a regular local volatility $\sigma(t, x)$.

- Prices adjustment is made through a least square minimization program,
- including a penalization term related to the local volatility regularity.

$$J(\alpha, \sigma) = \sum_{i,j} \omega_{i,j} \left(f(t_0, x_0, \phi_{i,j}, T_i, \sigma)) - C^{Obs}_{i,j} \right)^2 + \alpha ||\nabla \sigma||^2 \to \min_{\sigma}$$

- Existence and uniqueness are only partially solved

Asymptotic Behavior

Using Large Deviation Theory, Beresticky, Buscat (2001)

$$\Sigma^{\text{implied}}(K, t_0)^{-1} = \ln(\frac{K}{x_0})^{-1} \int_{x_0}^{K} \frac{d\xi}{\xi \sigma(\xi, t_0)}$$

Options on Several Assets I

Main objectif

- Option prices on every component are assumed to be calibrated from the data.
- Given the risk neutral cumulative function of the asset i, $F_i(x)$ and the fact that $F_i(X^i_T)$ is uniformly distributed, the problem may be reduced to work on $[0,1]^n$

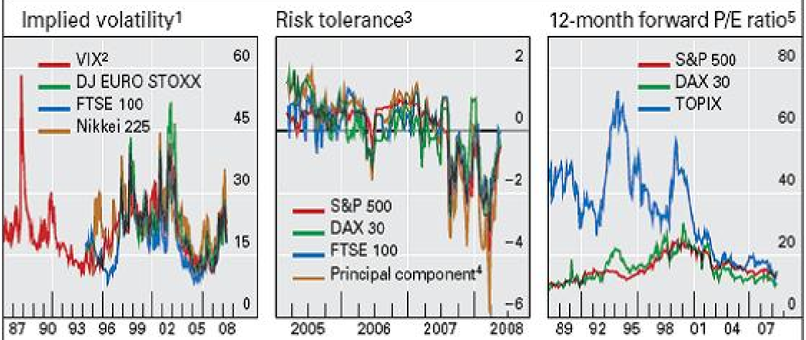
Static point of view **Def** A copula is the cumulative distribution function of a measure on $[0,1]^n$ with uniform marginal distributions.

- Non parametric problem, very complicated to calibrate
- Market uses in general Gaussian copula with few parameters (only the correlation if $n=2$

Quid of the dynamic hedging ?

Stock Volatility During the crisis

Equity market volatility and valuations



[1] Annualised volatility implied by the price of at-the-money call option contracts on stock market indices; monthly averages, in per cent. [2] Based on S&P 500; prior to 1990, based on S&P 100. [3] Derived from the differences between two distributions of returns, one implied by option prices, the other based on actual returns estimated from historical data; weekly averages of daily data. [4] First principal component of risk tolerance indicators for the S&P 500, DAX 30 and FTSE 100. [5] Ratio of the stock price and 12-month forward earnings per share.

Sources: Bloomberg; I/B/E/S; BIS calculations. Graph VI.12

Value At Risk I

When it is not possible to replicate some pay-off,(incomplete market) the trader has to measure his market residual risk exposure.

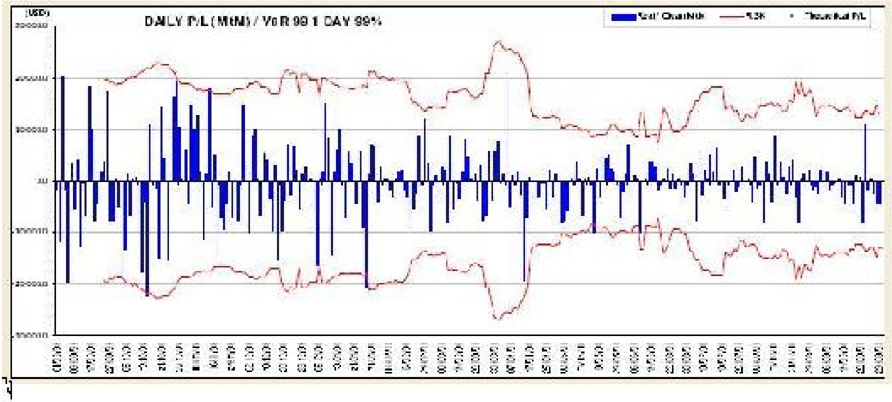
- The traditional measure is the variance of the replicating error. But a new criterion, taking into account extreme events, is now used.
- The VaR criterion, corresponding to the maximal level of losses acceptable with a given probability (95%), has taken considerable importance for several years.

$$VaR_{\varepsilon}(X) = \inf\{k : \mathbb{P}(X + k < 0) \leq \varepsilon\}.$$

Value At Risk II

- Regulation Authorities have required a daily VaR computation of the global risk portfolio from financial institutions. Such a measure affects the reserve a bank has to hold to face market risks.
- Processes with heavy tail distributions (Lévy processes with non-continuous paths) are now used in portfolio optimization and stress-testing.

Daily VaR on a aggregated portfolio



Among Strategic Mathematics Problem

- Renewed interest in Levy processes
- Renewed interest in fractional Brownian motion

- Fundamental Theorem of Asset Pricing
- Optional decomposition of supermartingale
- Coherent and convex risk measures
- Change of numéraire and market models for interest rates
- Evolution of the yield curve on a manifold

- Monte Carlo methods for prices and their derivatives, since prices are also given as expectation
- Monte Carlo methods for American options and controlled problems, by using non linear BSDE's

Why these methods do not prevent the crisis ?

- In credit derivatives market, only static models were used, too simple
- In incomplete market, it is difficult to estimate the residual risk
- Daily risk management, by delta hedging or value at risk has to be completed by different indicators relative to different time scales
- Liquidity risk in particular are not captured
- counterpart risk is minimized
- systemic risk has been undervalued
- Other indicators, as the size of the positions (2000 Billions of subprime) exist outside of math's criterium.

Demand for technology

- Wall street is exploring the use of graphics processing units found in video games to speed up options analytic and other math-intensive applications
- All developments in Monte Carlo simulation are made efficient by the new power of computer
- New developments in algorithmic trading, where engine is used to place trades using an electronic order book

Demand for people

- First master's programs in quantitative finance were founded fewer than 15 years ago (in Paris in 1990)
- Approximately 75 quantitative finance programs worldwide, (approx 2.000 quants students graduate annually...)(100 in Paris VI Master's Program)
- More diversified jobs : in regulation, insurance and accounting firms... in today's distressed market

Market are not like physical systems

At least three common behaviors cannot be with (simple) maths :

- Intentionality of human actions/reactions
- Subjective notion of risk
- Strategic Behaviors
- Asymmetric information

So Game theory for example is to be taken into consideration, which is practically more difficult to deal with.

Conclusion

The end of a bubble : yes ! but not the end of mathematics in finance.

- Mathematicians bring rigor to the party, and rigor is a critical part of quantitative finance, and risk management
- More demand for quantitative risk management
- Technology evolves quickly in Financial markets

Still remember that in social sciences there are not truly reproducible situations. So maths can only yield to partial representation of the complex reality

APPENDIX

Monetary Risk Measure

Monetary Convex Risk Measure I

Academics have debated on the VaR significance as a risk measure.
For instance, its non sub-additive property enables banks to play with the creation of subsidiaries. (Revision on the criterion ?)

Monetary Convex Risk Measure II

Definition (Characterization of Risk Measure)

The functional ρ is a convex risk measure if it satisfies the following properties :

1. Convexity and Decreasing monotonicity]
2. Translation invariance : $\forall m \in \mathbb{R}, \quad \rho(X+m) = \rho(X) - m.$
   "+" technical property of decreasing continuity from below to work with probability measures later (dual representation) : $\Psi_n \nearrow \Psi \quad \Longrightarrow \quad \rho\big(\Psi_n\big) \searrow \rho(\Psi).$
3. Coherent : $\forall \lambda > 0, \quad \rho(\lambda X) = \lambda \rho(X).$

Duality and penalty function I

The convexity of the framework leads to an "explicit" representation in terms of scenarii (Delbaen, Foellmer-Schied)

Theorem

There exists a penalty function α *taking values in* $\mathbb{R} \cup \{+\infty\}$ *s.t.*

$$\forall \Psi \in \mathcal{X}, \quad \rho(\Psi) = \sup_{\mathbb{Q} \in \mathcal{M}_{1,f}} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \alpha(\mathbb{Q})\}$$

$$\forall \mathbb{Q} \in \mathcal{M}_{1,f}, \quad \alpha(\mathbb{Q}) = \sup_{\Psi \in \mathcal{X}} \{\mathbb{E}_{\mathbb{Q}}(-\Psi) - \rho(\Psi)\}$$

Example (Entropic risk measure)

$$e_\gamma(X) = \gamma\mathbb{E}\big(\exp(-\frac{1}{\gamma}X)\big) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \big(\mathbb{E}_\mathbb{Q}(-X) - \gamma h(\mathbb{Q}|\mathbb{P})\big)$$

where h is the entropic function

$$h(\mathbb{Q}|\mathbb{P}) = \mathbb{E}_\mathbb{P}\big(\tfrac{d\mathbb{Q}}{d\mathbb{P}}\ln[\tfrac{d\mathbb{Q}}{d\mathbb{P}}]\big)$$

$$\text{if } \mathbb{Q} \ll \mathbb{P}, \quad +\infty \text{ otherwise}$$